

RELATIONS BETWEEN CERTAIN CONTINUOUS TRANSFORMATIONS
OF SETS

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RELATIONS BETWEEN CERTAIN CONTINUOUS TRANSFORMATIONS OF SETS

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In a paper on arc-preserving transformations¹ G. T. Whyburn has shown that under certain conditions a transformation is a homeomorphism on each cyclic element of a compact locally connected continuum. The principal result is obtained by using a transformation which is arc-preserving and irreducible. After studying Whyburn's theorems, the writer investigated the problem of finding conditions for a homeomorphism on the whole of any compact locally connected continuum and also of finding other conditions for a homeomorphism on the cyclic elements. In working with continua other than cyclic elements, it was found necessary to define some new transformations and to impose conditions on the sets obtained by the transformations. After true arc-preserving and strongly irreducible transformations were defined, it was found that one of Whyburn's proofs could be modified to give a proof of conditions for a homeomorphism on any compact locally connected continuum. The writer has also defined a tree-preserving transformation and has found additional conditions which will make it a homeomorphism in both cases, first on the cyclic elements and then on the whole of any compact locally connected continuum. To do this, true tree-preserving and strongly monotonic transformations have been defined. After these various transformations were studied, it was found that certain relations existed between them, and theorems about some of these relations are also proved in this paper.

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We shall assume throughout that our space is metric and compact. A knowledge of the theory of cyclic elements will also be assumed. Definitions of and theorems about cyclic elements may be found in an expository paper by Kuratowski and Whyburn.²

The symbol $\Phi(A)$ will mean a transformation Φ on the set A , where A is a compact locally connected continuum. All of the transformations which are considered are assumed to be single-valued and continuous. For a single-valued transformation continuity is equivalent to the property that a closed set comes

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¹ G. T. Whyburn, *American Journal of Mathematics*, vol. 58(1936), pp. 305-312.

² C. Kuratowski and G. T. Whyburn, *Fundamenta Mathematicae*, vol. 16(1930), pp. 305-331.

from a closed set, and, since compactness is assumed, it follows that a closed set goes into a closed set. These facts can be easily verified and have been found useful in the proofs of some of the theorems.

The following types of transformations are used in the proofs of the theorems in this paper: arc-preserving, true arc-preserving, tree-preserving, true tree-preserving, homeomorphic, contracting, irreducible, strongly irreducible, monotonic, and strongly monotonic. Four of these have been studied previously, but the remaining six, viz., the true arc-preserving, tree-preserving, true tree-preserving, contracting, strongly irreducible, and strongly monotonic transformations are defined for the first time in this paper. The definitions of these transformations will now be given and the reasons for introducing the new ones will be stated.

If $\Phi(A) = B$, Φ is said to be *arc-preserving* if the image of every simple arc in A is either a simple arc or a single point in B .³ A *true arc-preserving* transformation is defined so as to eliminate the possibility of an arc being carried into a point, that is, the image of every non-degenerate arc in A is a non-degenerate arc in B .

To obtain a transformation which is a little more general, a *tree-preserving* transformation is defined as follows. If $\Phi(A) = B$, Φ will be said to be *tree-preserving* if the image of every subset of A which is a tree is either a non-degenerate tree or a single point in B . As in the case of the arc-preserving transformation, in order to eliminate the possibility of a tree in A being carried into a single point, a *true tree-preserving* transformation is defined as one in which the image of every non-degenerate tree in A is a non-degenerate tree in B .

The definition of a homeomorphic or topological transformation is well known and will be omitted. A new transformation is introduced which only requires that when $\Phi(A) = B$, B shall be homeomorphic with a subset of A . In this case Φ is called a *contracting* transformation.

Whyburn has used the idea of the irreducibility of a transformation in his paper on arc-preserving transformations. Given $\Phi(A) = B$, he defines Φ as an *irreducible* transformation if no proper subcontinuum of A maps onto all of B .⁴ In order to extend theorems for cyclically connected sets to the case of any locally connected continuum, it has been found necessary to define a new transformation in which subcontinuum of A is replaced by any closed subset of A . That is, the condition that this subset of A be connected has been removed. Accordingly, given $\Phi(A) = B$, Φ is defined as a *strongly irreducible* transformation if no proper closed subset of A maps onto all of B .

In a paper on the structure of continua,⁵ Whyburn states a definition of a monotonic transformation. Given $\Phi(A) = B$, Φ is defined as a *monotonic* transformation if $\Phi^{-1}(b)$ is connected for every $b \in B$. $\Phi^{-1}(b)$ is used to represent

³ G. T. Whyburn, loc. cit.

⁴ G. T. Whyburn, loc. cit.

⁵ G. T. Whyburn, Bulletin of the American Mathematical Society, vol. 42(1936), pp. 49-71.

the set of all points x of A such that $\Phi(x) = b$, where $b \in B$. This idea has been used previously by C. B. Morrey.⁶ A new definition has been introduced because the condition that $\Phi^{-1}(b)$ be connected was found not to be sufficient in the case of locally connected continua not cyclically connected. A *strongly monotonic* transformation provides not only that $\Phi^{-1}(b)$ be connected, but that it be arcwise connected.

A. Sufficient conditions for a homeomorphism. In his paper on arc-preserving transformations⁷ Whyburn proved: "If A is a locally connected continuum and $\Phi(A) = B$ is irreducible and arc-preserving, then Φ is a homeomorphism on each true cyclic element of A on which Φ is not constant". In order to obtain conditions which will make Φ a homeomorphism on all of A , the true arc-preserving and strongly irreducible transformations have been defined.

The need of these stronger conditions may be seen from the following examples. First let A be a set which is the sum of three arcs B , C , and D such that B joins c and d , interior points of C and D , respectively. Let Φ be a transformation which is arc-preserving and irreducible on A and such that $\Phi(B) = \Phi(c) = \Phi(d)$ and $\Phi(C) \cdot \Phi(D) = \Phi(B)$. It is evident that Φ is not a homeomorphism on A , but since $\Phi(B)$ is a single point, Φ is not *true arc-preserving*. Next let A be the sum of three arcs B , C , and D such that $B \cdot C = x$, $C \cdot D = y$, and $B \cdot D = 0$. Let Φ be a transformation which is true arc-preserving and irreducible on A and homeomorphic on B , C , and D , and such that $\Phi(B) \cdot \Phi(D) = \Phi(C)$. If c is any interior point of $\Phi(C)$, $\Phi^{-1}(c)$ contains three points of A . Therefore Φ is not a homeomorphism on A . But since $\Phi(A)$ can be obtained from a closed subset of A , namely, $B + D$, Φ is not *strongly irreducible*.

THEOREM 1. *If A is a locally connected continuum and Φ is true arc-preserving and strongly irreducible on A , then Φ is a homeomorphism on A .⁸*

Let $\Phi(A) = B$. If Φ is not a homeomorphism, then for some $b \in B$, $\Phi^{-1}(b)$ contains at least two points x and y . Since A is arcwise connected, there is an arc xy in A , and since Φ is true arc-preserving, $\Phi(xy)$ is a true arc. Select a point $p \neq x$ or y so that $\Phi(p)$ is an endpoint of $\Phi(xpy)$. Since A is locally connected, U_x and U_y can be selected as arcwise connected neighborhoods of x and y , respectively, and such that $U_x \cdot U_y = 0$. Since under a single-valued continuous transformation a closed set comes from a closed set, $\Phi^{-1}(p)$ is a closed set. Therefore U_x and U_y can be selected so that $U_x \cdot \Phi^{-1}(p) = 0$ and $U_y \cdot \Phi^{-1}(p) = 0$. Since U_x is open, $A - U_x$ is closed. Since Φ is strongly irreducible, there is a point c in U_x such that $\Phi^{-1}[\Phi(c)] \subset U_x$, for otherwise the closed set $A - U_x$ would map onto all of B . Similarly there is a point d in U_y such that $\Phi^{-1}[\Phi(d)] \subset U_y$. Therefore $\Phi(c) \neq \Phi(d)$. U_x being arcwise connected, there

⁶ C. B. Morrey, American Journal of Mathematics, vol. 57(1935), pp. 17-50.

⁷ G. T. Whyburn, loc. cit.

⁸ The proof of this theorem is a modification of the proof of the theorem by Whyburn which is stated above.

is an arc from c to x in U_x . Let q be the first point on this arc from c to x which is also on the arc xpy . Similarly let r be the first point on an arc from d to y in U_y which is also on the arc xpy .

Let $M = cq + qp + pr + rd$, which is an arc in A . If Φ is true arc-preserving, $\Phi(M)$ is an arc, and one of the three distinct points $\Phi(c)$, $\Phi(d)$, and $\Phi(p)$ must separate the other two on this arc. $\Phi(qpr)$ is a subarc of $\Phi(xy)$ and this contains a subarc N from $\Phi(q)$ to $\Phi(r)$ which does not contain $\Phi(p)$, for $\Phi(p)$ is an endpoint of $\Phi(xy)$ and since $U_x \cdot \Phi^{-1}(p) = 0$ and $U_y \cdot \Phi^{-1}(p) = 0$, $\Phi(q) \neq \Phi(p) \neq \Phi(r)$. Since $\Phi(cq) \cdot \Phi(p) = 0$ and $\Phi(dr) \cdot \Phi(p) = 0$, $\Phi(cq) + N + \Phi(dr)$ is a connected subset of $\Phi(M)$ which does not contain $\Phi(p)$. Therefore $\Phi(p)$ does not separate $\Phi(c)$ and $\Phi(d)$. $\Phi(cq + qp)$ is a connected subset of $\Phi(M)$ which does not contain $\Phi(d)$, and therefore $\Phi(d)$ does not separate $\Phi(p)$ and $\Phi(c)$. Also $\Phi(dr + rp)$ is a connected subset of $\Phi(M)$ which does not contain $\Phi(c)$ and therefore $\Phi(c)$ does not separate $\Phi(p)$ and $\Phi(d)$. Since no one of the three points separates the other two on $\Phi(M)$, $\Phi(M)$ is not an arc. But this is a contradiction, and therefore Φ is a homeomorphism on A .

In order to obtain similar theorems by replacing an arc-preserving by a tree-preserving transformation on A , it was found necessary to place a condition on $\Phi(A)$. If $\Phi(A)$ does not contain a free arc,⁹ it is possible to have Φ tree-preserving and irreducible on A and yet not a homeomorphism. This is illustrated by the following example where A is a circle and $\Phi(A) = B$ is a universal tree of order four.

The tree of order four which is described here is slightly different from the one described by K. Menger.¹⁰ Let C be the portion of the (r, θ) -plane which is bounded by $r = 1$. Let I be that portion of C which lies in the first quadrant. Let L be the line $\theta = 45^\circ$ and let I_a and I_b be the two portions of C into which I is divided by L . Let S be a segment of L with $(0, 0^\circ)$ as one endpoint and such that all of its other points lie within I . Select a countable set of points $\{b_n\}$ dense on S and not including the endpoints of S . At b_1 erect two perpendiculars, S_a^1 and S_b^1 , such that all of their points except b_1 lie in the interior of D_a^1 and D_b^1 , triangles with b_1 as one vertex which lie in I_a and I_b , respectively, and are of diameter $< \frac{1}{2}$. At b_2 erect two perpendiculars, S_a^2 and S_b^2 , such that all of their points except b_2 lie in the interior of D_a^2 and D_b^2 , triangles with b_2 as one vertex which lie in I_a and I_b , respectively, are of diameter $< \frac{1}{3}$, and are such that $D_a^1 \cdot D_a^2 = 0$ ($\alpha = a, b$). Continuing in this manner for all the b_n 's, in general erect two perpendiculars, S_a^i and S_b^i , at b_i such that all of their points except b_i lie in the interior of D_a^i and D_b^i , triangles with b_i as one vertex which lie in I_a and I_b , respectively, are of diameter $< 1/(i + 1)$, and are such that $D_a^i \cdot \sum_{j=1}^{i-1} D_a^j = 0$. Let $T_I^1 = \sum_{\alpha, i} S_\alpha^i$.

On each S_α^i select a countable set of points dense on the segment and not

⁹ An arc xy of a set M is called a free arc if the arc $xy - (x + y)$ is an open subset of M . In other words, no interior point of the arc is a limit point of points of M not on the arc.

¹⁰ K. Menger, *Kurventheorie*, 1932, pp. 318-322.

including its endpoints, and at each of these points erect two perpendiculars, S_{ac}^{ij} and S_{ad}^{ij} (where the superscripts correspond to the numbering in the countable sets of points selected and the subscripts indicate in what portion of I the perpendicular lies), satisfying conditions like those for the perpendiculars to S , so that $T_I^2 = \sum S_{\alpha\beta}^{ij}$ ($\beta = c, d$) will not contain more than a finite number of arcs of diameter $> \epsilon$ for any ϵ .

This process can be continued to obtain $T_I^3, T_I^4, \dots$ so that there is no free arc in $\bar{B}_I$, where $B_I = S + \sum_{i=1}^{\infty} T_I^i$. Let $\bar{B}_{II}$, $\bar{B}_{III}$, and $\bar{B}_{IV}$ be similar sets constructed in the second, third, and fourth quadrants, respectively. Let $B = \sum_{i=1}^{IV} \bar{B}_i$. B is a universal tree of order four.

Let A be the circle $r = 1$, and let A be transformed into B in the following manner. Let Φ_0 be a transformation which carries $(1, 0^\circ)$, $(1, 90^\circ)$, $(1, 180^\circ)$, and $(1, 270^\circ)$ into $(0, 0^\circ)$ and such that each of the four arcs into which A is divided by these points is carried into a polygon P_i^1 ($i = I, \dots, IV$) lying within the part of C which is in the i -th quadrant, and such that a component of $B - (0, 0^\circ)$ is wholly in the interior of it. Let Φ_1 transform four points of P_I^1 into b_1 and carry the four arcs into which P_I^1 is divided by these points into four polygons P_{Ii}^2 ($i = 1, \dots, 4$) within P_I^1 and such that a component of $\bar{B}_I - b_1$ lies wholly within each, and let Φ_1 transform P_{II}^1, P_{III}^1 , and P_{IV}^1 in a similar way. Let Φ_2 be a transformation which carries four points of the P_{Ii}^2 in which b_2 lies into b_2 and the four arcs into four polygons as above, and let Φ_2 transform a P_{ij}^2 in each of the other quadrants in the same way. If this process is continued for each branch point of B , then $\Phi = \lim \Phi_i$ is a transformation of A into B which is tree-preserving and irreducible, but not a homeomorphism.

THEOREM 2. *If A is a locally connected continuum which is cyclically connected, and Φ is a tree-preserving and irreducible transformation on A , and $\Phi(A)$ contains a free arc, then Φ is a homeomorphism on A .*

Let $\Phi(A) = B$. If Φ is not a homeomorphism on A , then for some $b \in B$, $\Phi^{-1}(b)$ contains at least two points x and y . Let $v \neq b$ be an interior point of a free arc in $\Phi(A)$. Let v_1 be a point of $\Phi^{-1}(v)$. Since A is cyclically connected, there is an arc xv_1y in A . Since Φ is tree-preserving on A , $\Phi(xv_1y)$ is a tree. Let this be T . T contains $b = \Phi(x) = \Phi(y)$ and v and therefore an arc from b to v . Since v is an interior point of a free arc in $\Phi(A)$, there is a subarc pv of the arc bv such that p is an interior point of a free arc in T . Let p_1 be a point of $\Phi^{-1}(p)$ on the arc xv_1y .

Since $p_1 \neq v_1$, p_1 must be on the arc v_1x or the arc v_1y . Suppose p_1 is on the arc v_1y . Select a neighborhood U_{p_1} such that it contains no point of the arc v_1x . Since A being cyclically connected has no cut points, there is a $U_{p_1}^1 \subset U_{p_1}$ such that $A - U_{p_1}^1$ is connected.¹¹ Since Φ is irreducible on A , there is a point c_1 in $U_{p_1}^1$ such that $\Phi^{-1}[\Phi(c_1)] \subset U_{p_1}^1$, for otherwise $A - U_{p_1}^1$ would map onto all

¹¹ H. M. Gehman, Proceedings of the National Academy of Sciences, vol. 14(1928), pp. 431-433.

of $\Phi(A)$. Let the diameter of $U_{p_1}^1$ be ϵ and select a sequence of neighborhoods $\{U_{p_1}^i\}$ such that $U_{p_1}^i$ is of diameter $< \epsilon i^{-1}$, and select a sequence $\{c_i\}$ such that $\Phi^{-1}[\Phi(c_i)] \subset U_{p_1}^i$. Since p_1 is a limit point of $\{c_i\}$, p is a limit point of the corresponding sequence $\{c'_i\}$ in $\Phi(A)$. Since p is an interior point of a free arc, there is some c'_k on the arc bv such that $\Phi^{-1}(c'_k) \subset U_{p_1}^k$. But the arc v_1x is a subarc of the arc xy and therefore $\Phi(v_1x)$ is a subtree of T , and since $\Phi(v_1x)$ contains b and v , it also contains c'_k . Since v_1x contains a point of $\Phi^{-1}(c'_k)$, we have a contradiction, and therefore Φ is a homeomorphism on A in this case. The other case may be proved in the same way.

THEOREM 3. *If A is a locally connected continuum, and Φ is a true tree-preserving and strongly irreducible transformation on A , and every arc of $\Phi(A)$ contains a free arc, then Φ is a homeomorphism on A .*

Let $\Phi(A) = B$. If Φ is not a homeomorphism on A , then for some $b \in B$, $\Phi^{-1}(b)$ contains at least two points x and y . Since A is a locally connected continuum, there is an arc xy in A . Since Φ is true tree-preserving, $\Phi(xy)$ is a tree. Let $\Phi(xy) = T$. Let v be any point of T other than b , and let v_1 be a point of $\Phi^{-1}(v) \cdot (\text{arc } xy)$. Let p be some point of the arc bv in T distinct from v and b and on a free arc of $\Phi(A)$. This is possible since every arc of $\Phi(A)$ contains a free arc. Let p_1 be a point of $\Phi^{-1}(p) \cdot (\text{arc } xy)$. Then $p_1 \neq v_1$, for Φ is a single-valued transformation.

Since $p_1 \neq v_1$, p_1 must be on the arc v_1x or the arc v_1y . Suppose p_1 is on the arc v_1y . Let $U_{p_1}^1$ be a neighborhood of p_1 which contains no point of the arc v_1x . Since $U_{p_1}^1$ is open, $A - U_{p_1}^1$ is closed. Since Φ is strongly irreducible on A , there is a point c_1 in $U_{p_1}^1$ such that $\Phi^{-1}[\Phi(c_1)] \subset U_{p_1}^1$, for otherwise $A - U_{p_1}^1$ would map onto all of $\Phi(A)$. Let the diameter of $U_{p_1}^1$ be ϵ and select a sequence of neighborhoods $\{U_{p_1}^i\}$ such that $U_{p_1}^i$ is of diameter $< \epsilon i^{-1}$, and select a sequence $\{c_i\}$ such that $\Phi^{-1}[\Phi(c_i)] \subset U_{p_1}^i$. Since p_1 is a limit point of $\{c_i\}$ in A , p is a limit point of the corresponding sequence $\{c'_i\}$ in $\Phi(A)$. Since p is on a free arc, there is some c'_k on this arc. $\Phi^{-1}(c'_k) \subset U_{p_1}^k$. But the arc v_1x is a connected subset of the arc xy and therefore $\Phi(v_1x)$ is a connected subset of T , and since $\Phi(v_1x)$ contains b and v , it also contains c'_k . Since the arc v_1x contains a point of $\Phi^{-1}(c'_k)$, we have a contradiction, and therefore Φ is a homeomorphism on A .

This theorem without the condition that every arc of $\Phi(A)$ contains a free arc is not true. Let A be an arc ab and let c and d be any two distinct interior points of A in the order $acdb$. Let $\Phi(ac)$ and $\Phi(bd)$ be arcs having only $\Phi(c) = \Phi(d)$ in common, and let Φ be a homeomorphism on each of these arcs. Let $\Phi(cd) \cdot \Phi(ac + bd) = \Phi(c)$, where $\Phi(cd)$ is defined in the following manner. Let $\Phi_1(cd) = Y$ be the circle $r = 1$ in the example of the universal tree of order four described above, and let Φ_1 be such that for any $y \in Y$ except $\Phi(c) = \Phi(d)$, $\Phi_1^{-1}(y)$ contains only one point of the arc cd . Let Φ_2 be a transformation which carries Y into a universal tree just as Φ does in the example above. Let $\Phi = \Phi_2 \cdot \Phi_1$. Since $\Phi(cd)$ is a universal tree, it contains no free arc. Φ is tree-preserving on A , for since $\Phi(A)$ is a tree, every closed connected subset of $\Phi(A)$

is a tree, and Φ is strongly irreducible on A , for Φ is a homeomorphism on the arc ac and on the arc bd , and all of the arc cd except its endpoints is needed to obtain $\Phi(cd)$. But since $\Phi(A)$ is not an arc, Φ is not a homeomorphism on A .

In order to find other conditions for a homeomorphism on a locally connected continuum, a strongly monotonic transformation has been defined, for it was found necessary to have $\Phi^{-1}(b)$ not only connected but arcwise connected for every $b \in B$.

THEOREM 4. *If A is a locally connected continuum and Φ is a strongly monotonic and true tree-preserving transformation on A , then Φ is a homeomorphism on A .*

Let $\Phi(A) = B$. If Φ is not a homeomorphism on A , then, for some $b \in B$, $\Phi^{-1}(b)$ contains at least two points x and y . Since Φ is strongly monotonic on A , there is an arc xy in A such that $\Phi(xy) = b$. But since Φ is true tree-preserving on A , this is a contradiction, and therefore Φ is a homeomorphism on A .

This theorem is not true if Φ is strongly monotonic but simply tree-preserving instead of true tree-preserving on A . If A is an arc and $\Phi(A)$ is a single point, Φ is strongly monotonic and tree-preserving, but Φ is not a homeomorphism on A .

In Theorem 4, if A is hereditarily locally connected, strongly monotonic can be replaced by monotonic.

THEOREM 5. *If A is a hereditarily locally connected continuum and Φ is a monotonic and true tree-preserving transformation on A , then Φ is a homeomorphism on A .*

Let $\Phi(A) = B$. For any $b \in B$, $\Phi^{-1}(b)$ is closed, for a closed set comes from a closed set under a single-valued continuous transformation. Since Φ is monotonic on A , $\Phi^{-1}(b)$ is connected. Since $\Phi^{-1}(b)$ is closed and connected, it is a subcontinuum of A . Since A is hereditarily locally connected, any subcontinuum is arcwise connected. Therefore Φ is strongly monotonic on A , and by Theorem 4, Φ is a homeomorphism on A .

B. Other relations between transformations. In order to prove other relations between transformations, it has been found convenient to establish a few lemmas concerning a configuration which has been called a Y_n -set and to introduce a new subdivision of a tree.

A Y_n -set is defined as the sum of n arcs, $n \geq 3$, having a common endpoint and such that this endpoint is the only point common to any two of the arcs. The arcs will be called the *branches* of the set, and their common endpoint will be called the *center* of the set. Thus $Y_3 = A + B + C$ with center x will mean that A , B , and C are three arcs having x as a common endpoint and $A \cdot B = B \cdot C = A \cdot C = x$.

LEMMA 1. *Given $Y_3 = A + B + C$ with center x . If Φ is arc-preserving on Y_3 , then $\Phi(Y_3)$ is an arc or a single point, or $\Phi(Y_3) = \Phi(A) + \Phi(B) + \Phi(C)$, where $\Phi(A)$, $\Phi(B)$, and $\Phi(C)$ are three arcs having $\Phi(x)$ as common endpoint and $\Phi(A) \cdot \Phi(B) = \Phi(B) \cdot \Phi(C) = \Phi(A) \cdot \Phi(C) = \Phi(x)$.*

I. $\Phi(Y_3)$ will be an arc or a single point if $\Phi(A) \subset \Phi(B + C)$. $B + C$ is an arc and therefore since Φ is arc-preserving, $\Phi(B + C)$ is an arc or a single point. But as $\Phi(A) \subset \Phi(B + C)$, $\Phi(Y_3) = \Phi(B + C)$.

II. If $\Phi(Y_3)$ is not an arc or a single point, it follows from I that $\Phi(B)$ is not a subset of $\Phi(C)$ and $\Phi(C)$ is not a subset of $\Phi(B)$, and therefore $\Phi(B) \cdot \Phi(C)$ is a single point, or an arc from some point s to some point t in the arc $\Phi(B + C)$, where s and t are endpoints of the arcs $\Phi(C)$ and $\Phi(B)$, respectively. Since $B \supset x$ and $C \supset x$, $\Phi(B) \cdot \Phi(C) \supset \Phi(x)$.

If $\Phi(B) \cdot \Phi(C)$ is an arc st , let r be the other endpoint of $\Phi(C)$. From I we know that $\Phi(A)$ is not a subset of $\Phi(B + C)$, i.e., there is a point $a \in A$ such that $\Phi(a)$ is not an element of $\Phi(B + C)$. There is an arc ax in A and, since Φ is arc-preserving, $\Phi(ax)$ is an arc in $\Phi(Y_3)$. Let p be the first point that this arc has in common with $\Phi(B + C)$. Since $\Phi(A + C)$ is an arc, p must be an endpoint of $\Phi(C)$. If $p = r$, then $\Phi(C) \subset \Phi(A + B)$ and $\Phi(Y_3)$ is an arc by I. If $p = s$ then $\Phi(A + B)$ is not an arc unless $s = t = \Phi(x)$. Therefore, since Φ is arc-preserving and $\Phi(A + B)$ must be an arc, $\Phi(B) \cdot \Phi(C) = \Phi(x)$ and $\Phi(x)$ is a common end-point of $\Phi(B)$ and $\Phi(C)$.

Similarly it can be shown that $\Phi(A) \cdot \Phi(C) = \Phi(x)$ and $\Phi(A) \cdot \Phi(B) = \Phi(x)$, and $\Phi(x)$ is the common endpoint of any two of the arcs.

LEMMA 2. *Given a Y_n -set with center x , if Φ is arc-preserving on Y_n , then $\Phi(Y_n)$ is an arc or a single point, or $\Phi(Y_n)$ is the sum of the transforms of k of the branches of Y_n ($3 \leq k \leq n$) having $\Phi(x)$ as common endpoint and such that $\Phi(x)$ is the only point common to any two of them.*

Let $Y_n = M_1 + M_2 + \dots + M_n$. If there are two branches of Y_n , M_{i1} and M_{i2} , such that $\Phi(M_{i1} + M_{i2}) = \Phi(Y_n)$, then since $M_{i1} + M_{i2}$ is an arc and Φ is arc-preserving, $\Phi(Y_n)$ is an arc or a single point. If $\Phi(Y_n) = \Phi(M_{i1} + M_{i2} + \dots + M_{ik})$, where $k \geq 3$, and no $\Phi(M_{ij}) \subset \Phi(M_{i1} + M_{i2} + \dots + M_{ik} - M_{ij})$, then by repeated applications of Lemma 1 we may show that $\Phi(x)$ is the common endpoint of $\Phi(M_{i1})$, $\Phi(M_{i2})$, $\dots$, $\Phi(M_{ik})$ and is the only point common to any two.

COROLLARY 1. *Given $Y_n = M_1 + M_2 + \dots + M_n$ with center x . If Φ is arc-preserving on Y_n , and $\Phi(Y_n)$ is not an arc (or a single point), then*

- (a) $\Phi(M_i)$ is an arc with $\Phi(x)$ as endpoint [or $\Phi(M_i) = \Phi(x)$]; and
- (b.1) if $\Phi(M_i) \cdot \Phi(M_j) = \Phi(x)$ for every $M_j \neq M_i$, then $\Phi(M_i) \cdot \Phi(Y_n - M_i) = \Phi(x)$, where $N_i = M_i - x$, or
- (b.2) if $\Phi(M_i)$ and $\Phi(M_j)$ have points other than $\Phi(x)$ in common, either $\Phi(M_i) \subset \Phi(M_j)$ or $\Phi(M_j) \subset \Phi(M_i)$.

Remark. Lemma 2 and Corollary 1 are true for x a point of order ω . If the center is a point of order ω , the set will be called a Y_ω -set.

LEMMA 3. *If A is a tree, then $A = \sum_{s=1}^{\infty} A_s$ where*

- (1) A_1 is any arc whose endpoints are endpoints of A ;
- (2) for $s > 1$, $A_s = \sum_{j=1}^{n_s-2} D_{sj}$, where D_{sj} is an arc one endpoint of which is an endpoint of A , and the other endpoint is a point of order n_i and this is the only point which D_{sj} and $\sum_{i=1}^{s-1} A_i$ have in common, and the only point which two of the D_{sj} 's have in common;

(3) $A = \sum_{s=1}^{\infty} A_s$ consists entirely of endpoints of A .

Let $A_1 = L_1$ be some arc joining two endpoints of A . Select the component of $A - L_1$ which has the largest diameter. Since there cannot be more than a finite number of components of diameter $> \delta$ for any δ , the largest one can be selected. Let b_1 be the limit point which this component has on L_1 . Let the order of b_1 be n_1 . (If b_1 is a point of increasing order, n_1 will be ω , and there will be a countable infinity of components.) There will be $n_1 - 2$ components, C_j , of $A - L_1$ such that $\overline{C_j \cdot L_1} = b_1$. Since $C_j + b_1$ is closed and bounded, there is a point q such that $\rho(b_1, q) \geq \rho(b_1, r)$, where r is any other point of C_j . There is an arc D_{2j} with b_1 and an endpoint of A as endpoints and containing q . Select an arc in this manner from each C_j and let the sum of these arcs be A_2 ; i.e., $A_2 = \sum_{j=1}^{n_1-2} D_{2j}$. Let $L_2 = A_1 + A_2$.

Select the component of $A - L_2$ which has the largest diameter as above, and let b_2 be the limit point which it has on L_2 . Let the order of b_2 be n_2 . Select an arc in the manner described above from each of the $n_2 - 2$ components of $A - L_2$ having b_2 as a limit point. Let the sum of these arcs be A_3 and let $L_3 = \sum_{s=1}^3 A_s$. In general b_i is selected so that it is the limit point on L_i of as large a component of $A - L_i$ as possible. A_{i+1} is selected so that it contains one arc, as large as possible, from each component of $A - L_i$ which has b_i as a limit point. $L_j = \sum_{s=1}^j A_s$.

Any point p which is not an endpoint of A is an interior point of some arc uv .¹² Let the diameter of the arc up be ϵ . The A_s 's were selected in such a way that for any ϵ and for s_1 sufficiently large, no component of $A - \sum_{t=1}^{s_1} A_t$ is of diameter $> \epsilon$. Therefore some point e of the arc up belongs to $\sum_{t=1}^{s_1} A_t$. Similarly some point f of the arc vp belongs to $\sum_{t=1}^{s_2} A_t$ for s_2 large enough. Therefore both e and f belong to $\sum_{t=1}^s A_t$ for s equal to the larger of s_1 and s_2 . Since there is just one arc from e to f , this is in $\sum_{t=1}^s A_t$ and since this arc contains p , p is a point of $\sum_{t=1}^s A_t$ for s large enough. This leads to the conclusion that $A = \sum_{s=1}^{\infty} A_s$ consists entirely of endpoints of A .

Since each point of $A = \sum_{s=1}^{\infty} A_s$ is an endpoint and all endpoints are limit points of non-endpoints, $A = \sum_{s=1}^{\infty} A_s$.

¹² G. T. Whyburn, Transactions of the American Mathematical Society, vol. 29(1927), p. 385.

THEOREM 6. *If D is a locally connected continuum and Φ is arc-preserving on D , then Φ is tree-preserving on D .*

Although this result is one that might be expected intuitively, we have not been able to find a short method of proof. As the proof is long and detailed, an outline of the method used will be given first. We wish to show that if A is a tree, and $\Phi(A) = B$, and Φ is arc-preserving on A , then B is a tree. This is done by using three monotonic transformations, Φ_1 , Φ_2 , and Φ_3 . We let $\Phi_1(A) = A'$, $\Phi_2(A') = A''$, and $\Phi_3(A'') = A'''$. It is shown that A' , A'' , and A''' are trees. Φ is defined on A' , A'' , and A''' in such a way that $\Phi(A') = \Phi(A'') = \Phi(A''') = B$. Finally, A''' is obtained in such a way that Φ is a homeomorphism on A''' and this leads to the desired result that B is a tree. If we use these three monotonic transformations, sets which have certain desired properties are obtained. Φ_1 produces a set A' on which Φ is true arc-preserving. Φ_2 produces a set A'' such that if $z \in B$ has two distinct inverses on A'' , they are both on the same free arc. Φ_3 produces a set on which Φ is true arc-preserving and strongly irreducible, that is, as was shown in Theorem 1, a set on which Φ is a homeomorphism.

I

Let A be any subset of D which is a tree and let $\Phi(A) = B$.

Since Φ is not necessarily true arc-preserving on A , there may be arcs $\{\alpha_i\}$ such that $\Phi(\alpha_i)$ is a single point. Let $\{\beta_i\}$ be the set of components of $\sum \alpha_i$. Let Φ_1 be a monotonic transformation which is constant on each β_i , but not on any arc of $A - \sum \beta_i$. Let $\Phi_1(A) = A'$. Since under any monotonic transformation, the image of a tree is a tree,¹³ A' is a tree. Let $\Phi_1(\beta_i) = k_i$. Let $\{C_i\}$ be the components of $A - \sum \beta_i$. As Φ_1 is a monotonic transformation, connectedness is an invariant under Φ_1^{-1} ,¹⁴ and therefore $\Phi_1(C_i) \cdot \Phi_1(C_j) = 0$. If for any point $x \in \Phi_1(C_i)$, $\Phi_1^{-1}(x)$ contains two distinct points x^{-1} and x^{-2} , $\Phi_1^{-1}(x)$ contains the arc from x^{-1} to x^{-2} in A , as Φ_1 is monotonic, but since Φ_1 is not constant on any arc of C_i , this is a contradiction. Therefore $\Phi_1^{-1}(x)$ is a single point of A . If we define Φ on A' so that $\Phi(k_i) = \Phi(\beta_i)$ and $\Phi(x)$ for $x \in \Phi_1(A - \sum \beta_i)$ so that $\Phi(x) = \Phi[\Phi_1^{-1}(x)]$, then $\Phi(A') = B$ and Φ is true arc-preserving on A' .

II

(A) Let $\{b_i^0\}$ be the branch points of A' such that $\Phi(b_i^0)$ is a point of order one or two in B . Let $\{b_i'\}$ be the branch points of A' such that $\Phi(b_i')$ is a point of order greater than two in B . Let M_{ij} be the arc from b_i' to b_j' . Let $M = \sum M_{ij}$. Since M is a closed subset of A' and A' is a continuous curve, the number of components of $A' - M$ is countable.¹⁵ Let $\{S_{ki}\}$ be the set of these

¹³ C. Kuratowski, *Fundamenta Mathematicae*, vol. 11(1928), p. 182.

¹⁴ G. T. Whyburn, *American Journal of Mathematics*, vol. 56(1934), p. 295.

¹⁵ R. L. Wilder, *Fundamenta Mathematicae*, vol. 7(1925), p. 360.

components and let s_k be the limit point of S_{ki} in M . Each s_k must be a branch point of A' or a point of $M - \sum M_{ij}$. Let s_k^0 represent an s_k which is a b_i^0 , let s_k' represent an s_k which is a b_i' , and let s_k'' represent an s_k which is a point of $M - \sum M_{ij}$.

Case 1. $S_{ki} + s_k^0$. This case is treated below in (B).

Case 2. $S_{ki} + s_k'$. For any $x \in S_{ki}$, by Corollary 1(a), the arc xs_k' goes into an arc with $\Phi(s_k')$ as an endpoint.

Case 3. $S_{ki} + s_k''$. Any s_k'' is a limit point of $\{b_i'\}$ and there is a subsequence of $\{b_i'\}$ which approaches s_k'' along an arc of M . Let $\{b_{ij}'\}$ be such a subsequence which is ordered on the arc. For any $x \in S_{ki}$, by Corollary 1(a), every arc xb_{ij}' goes into an arc with $\Phi(b_{ij}')$ as an endpoint. Since $\Phi(xb_{i,j+1}') \subset \Phi(xb_{ij}')$ and $\lim \Phi\{b_{ij}'\} = \Phi(s_k'')$, $\Phi(xs_k'')$ is an arc having $\Phi(s_k'')$ as an endpoint by the Cantor Product Theorem.

In Cases 2 and 3, let y be any point of S_{ki} distinct from x . Since A' is a tree, the arc xs_k and the arc ys_k have a subarc in common. As Φ is true arc-preserving on A' , $\Phi(xs_k)$ and $\Phi(ys_k)$ have points other than $\Phi(s_k)$ in common, and by Corollary 1(b.2), one is a subset of the other. As Φ on $S_{ki} + s_k$ is a continuous transformation on a tree, $\Phi(S_{ki} + s_k)$ is a compact locally connected continuum. Since $\Phi(S_{ki})$ contains no point of order greater than two, $\Phi(S_{ki} + s_k)$ is an arc or a simple closed curve. We next show that it is an arc.

Let E be the set of endpoints of S_{ki} . As E is separable we can let $E = \{x_j\} + Q$, where $\{x_j\}$ is a countable set and every point of Q is a limit point of $\{x_j\}$. If, for any x_j , $\Phi(x_js_k) = \Phi(S_{ki} + s_k)$, then since $\Phi(x_js_k)$ is an arc, $\Phi(S_{ki} + s_k)$ is an arc. In this case let N_i be the arc x_js_k . If $\Phi(x_js_k) \neq \Phi(S_{ki} + s_k)$ for any x_j , then since either $\Phi(x_js_k) \subset \Phi(x_js_k)$ or $\Phi(x_js_k) \subset \Phi(x_js_k)$, there is a subsequence $\{x_j'\}$ of $\{x_j\}$ which can be ordered so that $\Phi(x_j's_k) \subset \Phi(x_{j+1}'s_k)$ and such that for any x_j not in $\{x_j'\}$, $\Phi(x_js_k)$ is contained in $\Phi(x_j's_k)$ for some x_j' .

$$\lim \Phi(x_j's_k) = \lim \sum \Phi(x_j's_k) = \lim \sum \Phi(x_js_k) = \Phi(S_{ki} + s_k).$$

One endpoint of $\Phi(x_j's_k)$ is $\Phi(s_k)$. Let e_j be the other endpoint. As $\{e_j\}$ is an ordered sequence on an arc, it has a sequential limit point p . Let e_j^{-1} be a point of $\Phi^{-1}(e_j)$ on the arc $x_j's_k$ and let $\{e_j^{-1}\}$ be a convergent subsequence of such points. Let $q = \lim \{e_j^{-1}\}$. If $\{e_j^{-1}\}$ converged to s_k , the diameter of the arcs $\{e_j^{-1}s_k\}$ would approach zero and the diameter of the arcs $\Phi\{e_j^{-1}s_k\}$ would approach zero, but $\Phi(e_j^{-1}s_k) = \Phi(x_j's_k)$ and $\Phi(S_{ki} + s_k) = \lim \Phi(x_j's_k)$ cannot be a single point since Φ is true arc-preserving on A' . Therefore $q \neq s_k$. For any e_j^{-1} there is a q_j such that arc $e_j^{-1}s_k + \text{arc } qs_k = \text{arc } q_js_k + \text{arc } q_j e_j^{-1} + \text{arc } q_j q$, where no two of these three arcs have more than q_j in common. Since A' is locally connected and $q = \lim \{e_j^{-1}\}$, the limit of the diameters of $\{q_j e_j^{-1}\}$ is zero. As q_j is on the arc qs_k , $q = \lim \{q_j\}$.

$$\lim \Phi\{e_j^{-1}s_k\} = \lim \Phi\{e_j^{-1}q_j\} + \lim \Phi\{q_js_k\} = \Phi(qs_k)$$

and

$$\lim \Phi\{e_j^{-1}s_k\} = \lim \Phi\{x_j's_k\} = \Phi(S_{ki} + s_k).$$

But $\Phi(qs_k)$ is an arc and therefore $\Phi(S_{ki} + s_k)$ is an arc. In this case let N_i be the arc qs_k .

Let $K_k = \sum N'_i$ where N'_i is an N_i which has an s'_k as an endpoint and such that $\Phi(N'_i) \cdot \Phi(M) = \Phi(s'_k)$. By Lemma 2 there is a $K'_k \subset K_k$ where $K'_k = \sum N''_i$ and the N''_i 's are N'_i 's such that $\Phi(N''_i) \cdot \Phi(N''_j) = \Phi(s'_k)$ and such that $\Phi(K'_k) = \Phi(K_k)$. Let H_k be any N_i which has an s''_k as an endpoint, and let H'_k be any H_k such that $\Phi(H'_k) \cdot \Phi(M) = \Phi(s''_k)$.

(B) $A' = M + \sum (S_{ki} + s^0_k) + \sum (S_{ki} + s'_k) + \sum (S_{ki} + s''_k)$. Let $V = M + \sum K'_k + \sum H'_k$. For some N_i , $\Phi(S_{ki} + s'_k) = \Phi(N_i)$. If this N_i is not in K'_k , then either $\Phi(N_i) \subset \Phi(N_j)$, where N_j does belong to K'_k , or $\Phi(N_i)$ and $\Phi(M)$ have points other than $\Phi(s_k)$ in common. Let M_{jk} be an M_{ij} such that $\Phi(M_{jk})$ and $\Phi(N_i)$ have points other than $\Phi(s_k)$ in common. By Corollary 1(b.2), either $\Phi(M_{jk}) \subset \Phi(N_i)$ or $\Phi(N_i) \subset \Phi(M_{jk})$. If $\Phi(M_{jk})$ is a proper subset of $\Phi(N_i)$, then $\Phi(b'_j)$ is on $\Phi(N_i)$. The point b'_j is the center of a Y_n -set, Y_j . Since $\Phi(b'_j)$ is the center of a Y_n -set, $\Phi(Y_j)$, there is a point $x \in \Phi(Y_j) - \Phi(M_{jk})$ and a point $y \in \Phi(N_i) - \Phi(M_{jk})$ such that $\Phi(b'_j)$, x , and y do not lie on an arc, but for any $x_1 \in \Phi^{-1}(x) \cdot Y_j$ and $y_1 \in \Phi^{-1}(y) \cdot N_i$, the points x_1 , b'_j , and y_1 lie on an arc and $\Phi(x_1 b'_j y_1)$ is an arc. Therefore in this case $\Phi(N_i) \subset \Phi(M_{jk})$. It follows that $\Phi(S_{ki} + s'_k)$ is contained in $\Phi(K'_k)$ or in $\Phi(M_{jk})$ for some M_{jk} . The same method of proof can be used to show that $\Phi(S_{ki} + s''_k)$ is contained in $\Phi(H'_k)$ or in $\Phi(M_{jk})$ for some M_{jk} . Any s^0_k which is not an s''_k on M is an interior point of an M_{ij} . Let W_k be any arc of $S_{ki} + s^0_k$ which has s^0_k as an endpoint. Since $\Phi(s^0_k)$ is a point of order less than three and Φ is true arc-preserving on A' , $\Phi(W_k + M_{ij})$ is an arc. If $\Phi(W_k)$ is not a subset of $\Phi(M_{ij})$, then an arc of $W_k + M_{ij}$ can be found which does not go into an arc, just as in the proof above. Therefore $\Phi(S_{ki} + s^0_k)$ is contained in $\Phi(M_{ij})$. It follows that $\Phi(V) \supset \Phi(A')$, but since V is a subset of A' , $\Phi(V) = \Phi(A')$.

Let $\{\gamma_i\}$ be the set of components of $A' - V$ and let g_i be the limit point of γ_i in $A' - V$. Let Φ_2 be a monotonic transformation which is constant on each γ_i , but not on any arc of V . Let $\Phi_2(A') = A''$. Since Φ_2 is monotonic and A' is a tree, A'' is a tree. Let $\Phi_2(\gamma_i) = j_i$. For any point $x \in \Phi_2(V)$, $\Phi_2^{-1}(x)$ is a single point of A' (see proof above that $\Phi_1^{-1}(x)$ is a single point of A). If we define Φ on A'' so that $\Phi(j_i) = \Phi(g_i)$ and $\Phi(x)$ for $x \in \Phi_2(V)$ so that $\Phi(x) = \Phi[\Phi_2^{-1}(x)]$, then $\Phi(A'') = B$.

III

Let X_i represent any branch of a K'_k and let Z_i represent any H'_k . Let $A'' = \sum G_i$ where each G_i is a $\Phi_2(M_{ij})$, a $\Phi_2(X_i)$, or a $\Phi_2(Z_i)$. It can be shown as follows that if G_i contains x but not y and G_j contains y but not x , then $\Phi(x) \neq \Phi(y)$. Let $x_1 = \Phi_2^{-1}(x) \cdot V$ and let $y_1 = \Phi_2^{-1}(y) \cdot V$.

(1) If M_{ij}^1 contains x_1 but not y_1 and M_{ij}^2 contains y_1 but not x_1 , the arc from x_1 to y_1 contains a b'_k . Select any b'_i and b'_j such that the arc $b'_i b'_k$ contains x_1 and the arc $b'_j b'_k$ contains y_1 . By Corollary 1(a), the images of these two arcs

each have $\Phi(b'_k)$ as an endpoint. If they have another point in common, by Corollary 1(b.2), one is a subset of the other. The points b'_i and b'_j are centers of Y_n -sets. Let these be Y_i and Y_j , respectively. Since $\Phi(Y_i)$ and $\Phi(Y_j)$ are Y_n -sets, there is a point $a \in \Phi(Y_i) - \Phi(M)$ and a point $c \in \Phi(Y_j) - \Phi(M)$ such that a , c , and $\Phi(b'_k)$ do not lie on an arc, but for any points $a_1 \in \Phi^{-1}(a) \cdot Y_i$ and $c_1 \in \Phi^{-1}(c) \cdot Y_j$, there is an arc $a_1 b'_k c_1$ and therefore $\Phi(a_1 b'_k c_1)$ must be an arc. As this is a contradiction, $\Phi(b'_i b'_k)$ and $\Phi(b'_j b'_k)$ must have only $\Phi(b'_k)$ in common, and $\Phi(x_1) = \Phi(x) \neq \Phi(y_1) = \Phi(y)$.

(2) If M_{ij}^1 contains x_1 but not y_1 and X_i contains y_1 but not x_1 , select any b'_j such that the arc from b'_j to b'_k , which is an endpoint of X_i , contains x_1 . By Corollary 1(a), $\Phi(b'_j b'_k)$ and $\Phi(X_i)$ each have $\Phi(b'_k)$ as an endpoint. If they have another point in common, by Corollary 1(b.2), one is a subset of the other, but there is an M_{ij}^2 which is a subset of the arc $b'_j b'_k$ and which has b'_k as an endpoint and the X_i 's were defined so that $\Phi(X_i) \cdot \Phi(M_{ij}^2) = \Phi(b'_k)$. Therefore in this case $\Phi(x_1) \neq \Phi(y_1)$.

(3) If M_{ij}^1 contains x_1 but not y_1 and Z_i contains y_1 but not x_1 , the proof that $\Phi(x_1) \neq \Phi(y_1)$ is the same as (2) with Z_i substituted for X_i .

(4) If X_i^1 contains x_1 but not y_1 and X_i^2 contains y_1 but not x_1 , let b'_i and b'_j be the b'_i 's which are endpoints of X_i^1 and X_i^2 , respectively. If $b'_i = b'_j$, $\Phi(X_i^1) \cdot \Phi(X_i^2) = \Phi(b'_i)$, for in this case X_i^1 and X_i^2 belong to the same K'_k . Suppose $b'_i \neq b'_j$ and $\Phi(x_1) = \Phi(y_1)$. There is an arc from $\Phi(x_1)$ to $\Phi(b'_i)$ in $\Phi(X_i^1)$ and an arc from $\Phi(y_1)$ to $\Phi(b'_j)$ in $\Phi(X_i^2)$ and therefore an arc from $\Phi(b'_i)$ to $\Phi(b'_j)$ in $\Phi(X_i^1) + \Phi(X_i^2)$. There is also an arc from $\Phi(b'_i)$ to $\Phi(b'_j)$ in $\Phi(M)$. From (2) we know that $\Phi(X_i^1) \cdot \Phi(M) = \Phi(b'_i)$ and $\Phi(X_i^2) \cdot \Phi(M) = \Phi(b'_j)$. Therefore $\Phi(x_1 b'_i) + \Phi(b'_i b'_j) + \Phi(b'_j y_1)$ contains a simple closed curve, but the image of the arc $x_1 b'_i b'_j y_1$ must be an arc. Therefore $\Phi(x_1) \neq \Phi(y_1)$.

(5) If Z_i^1 contains x_1 but not y_1 and Z_i^2 contains y_1 but not x_1 , the proof that $\Phi(x_1) \neq \Phi(y_1)$ is the same as the second part of (4) ($b'_i \neq b'_j$, for $Z_i^1 \cdot Z_i^2 = 0$) with Z_i substituted for X_i .

(6) If X_i contains x_1 but not y_1 and Z_i contains y_1 but not x_1 , the proof that $\Phi(x_1) \neq \Phi(y_1)$ is the same as the second part of (4) (for $b'_i \neq b'_j$) with X_i substituted for X_i^1 and Z_i substituted for X_i^2 and (3) used to show that $\Phi(Z_i) \cdot \Phi(M) = \Phi(b'_j)$.

Since these six cases cover all the possibilities, if for any $z \in B$, $\Phi^{-1}(z)$ contains two distinct points in $\sum G_i$, they both belong to the same G_i . Select any δ and any G_i . Let this be G_1 . If for any $z_j \in B$ there are two points u_1 and v_1 of $\Phi^{-1}(z_j) \cdot G_1$ such that $\rho(u_1 v_1) > \delta$ (where $\rho(u_1 v_1)$ is the diameter of the arc $u_1 v_1$), let c be any interior point of this arc. Let e_1 and e_2 be the endpoints of G_1 , where u_1 is on the arc $e_1 c$ and v_1 is on the arc $c e_2$. Let $\{x_i\}$ be the set of all points on the arc $e_1 u_1$ such that there is a point y_i on the arc $v_1 e_2$ such that $\Phi(x_i) = \Phi(y_i)$. From the continuity of the transformation, $\{x_i\}$ is closed and therefore there is a last point of $\{x_i\}$ on the arc from u_1 to e_1 . Let this be u' . Similarly there is a last point of $\{y_i\}$ such that $\Phi(y_i) = \Phi(u')$ on the arc from v_1 to e_2 . Let this be v' . Let the arc $u' v'$ be T_1 . Let T'_1 be T_1 minus its endpoints. If

for any $z_k \in B$ there are two points u_2 and v_2 of $\Phi^{-1}(z_k) \cdot (G_1 - T_1')$ such that $\rho(u_2 v_2) > \delta$, select an arc $u_2' v_2' = T_2$ in the same way that T_1 was selected. From the way in which T_1 and T_2 are determined, $T_1 \cdot T_2 = 0$. Continuing in this manner, we shall have a finite number of T_i 's $> \delta$, for G_1 cannot contain more than a finite number of mutually exclusive arcs of diameter $> \delta$. Next use $\frac{1}{2}\delta$ and G_1 minus the sum of the T_i 's previously defined. Continue this process with $\frac{1}{3}\delta, \frac{1}{4}\delta, \dots$. Repeat with $G_2 - G_1, \dots, G_n - \sum_{i=1}^{n-1} G_i, \dots$. After this has been completed for each G_i , $\sum G_i - \sum T_i'$ will not contain two points u_i and v_i such that $\Phi(u_i) = \Phi(v_i)$.

Let Φ_3 be a monotonic transformation which is constant on each T_i , but not on any arc of $A'' - \sum T_i$. Let $\Phi_3(A'') = A'''$. Since Φ_3 is monotonic and A'' is a tree, A''' is a tree. Let $\Phi_3(T_i) = h_i$. For any point $x \in \Phi_3(A'' - \sum T_i)$, $\Phi_3^{-1}(x)$ is a single point of A'' (see proof above that $\Phi_1^{-1}(x)$ is a single point of A). If we define Φ on A''' so that $\Phi(h_i)$ is the image of the endpoints of T_i and $\Phi(x)$ for $x \in \Phi_3(A'' - \sum T_i)$ so that $\Phi(x) = \Phi[\Phi_3^{-1}(x)]$, we can show as follows that $\Phi(A''') = B$. Let e_1 and e_2 be the endpoints of G_i and let $u_i v_i$ be any arc of G_i such that $\Phi(u_i) = \Phi(v_i)$. Let c be any interior point of the arc $u_i v_i$. Now, if $\Phi^{-1}(c) \cdot \text{arc } e_1 u_i = 0$, $\Phi(c)$ is not on $\Phi(e_1 u_i)$ and it must be on $\Phi(v_i e_2)$, and therefore $\Phi^{-1}(c) \cdot \text{arc } v_i e_2 \neq 0$. Therefore $\Phi(A''') = \Phi(A'' - \sum T_i) + \Phi \sum (T_i) = \Phi(A'') = B$.

IV

If, for any point $z \in B$, $\Phi_3^{-1}(z)$ contains two distinct points x and y in A''' , it follows from (1)-(6) above that they cannot belong to the images of two different G_i 's and from the definition of Φ_3 they cannot both belong to the same $\Phi_3(G_i)$ for any i . Therefore at least one of them must be a point of $\Phi_3(A'' - \sum G_i)$. Suppose x is such a point. Let x^{-1} be a point of $\Phi_3^{-1}(x) \cdot (A'' - \sum G_i)$ and let y^{-1} be a point of $\Phi_3^{-1}(y)$. As $\sum G_i$ contains all the points of $\Phi_2(\sum M_{ij} + \sum K'_k + \sum H'_k)$, and all the points of $\sum K'_k - \sum K'_k$ and of $\sum H'_k - \sum H'_k$ are in M , x^{-1} is a point of $\Phi_2(M - \sum M_{ij})$. Let $\Phi_2^{-1}(x^{-1}) \cdot V = x^{-2}$. Since x^{-2} is a limit point of a subsequence $\{b'_{ij}\}$ of $\{b'_i\}$ which approaches x^{-2} along an arc of M , x^{-1} is a limit point of $\Phi_2\{b'_{ij}\}$ which approaches x^{-1} along an arc of $\Phi_2(M)$. As A'' is a tree and x^{-1} is an endpoint of $\Phi_2(M)$, the arc $x^{-1} y^{-1}$ has a subarc in common with this arc of $\Phi_2(M)$ and there is a $\Phi_2(b'_i)$ on this subarc. Let $\Phi_2(b'_i) = b''_i$. Let I be the arc from x^{-1} to b''_i and let J be the arc from y^{-1} to b''_i . These arcs have subarcs G_i and G_j , respectively, with b''_i as an endpoint. If $\Phi(I) \cdot \Phi(J) \neq \Phi(b''_i)$, then by Corollary 1(b.2), either $\Phi(I) \subset \Phi(J)$ or $\Phi(J) \subset \Phi(I)$, but this is impossible for it has been shown above that if $G_i \cdot G_j = b''_i$, then $\Phi(G_i) \cdot \Phi(G_j) = \Phi(b''_i)$. Therefore Φ is a homeomorphism on A''' , and since A''' is a tree, B is a tree.

THEOREM 7. *If A is a tree and Φ is true arc-preserving on A , then Φ is contracting on A .*

Let $A' = A$, $\Phi_2(A') = A''$ and $\Phi_3(A'') = A'''$, where Φ_2 and Φ_3 are defined as in Theorem 6. Let $V \subset A'$, Φ on A'' , and Φ on A''' also be defined as in Theorem 6.

We know that Φ_2 is a homeomorphism on V and that $\Phi_2(V) = A''$. Therefore A'' is homeomorphic with V . The transformation Φ_3 is such that $\Phi_3(G_i - \sum T'_i)$ is an arc. Any two arcs are homeomorphic and therefore there is a transformation ψ such that $\psi(G_i) = \Phi_3(G_i - \sum T'_i)$ and such that ψ is a homeomorphism on each G_i . Since $\Phi_3(G_i) \cdot \Phi_3(G_j) = \Phi_3(G_i \cdot G_j)$, ψ is a homeomorphism on $A'' = \overline{\sum G_i}$. As $\psi(A'') = A'''$, A''' is homeomorphic with A'' . As Φ is a homeomorphism on A''' , B is homeomorphic with A''' . Since B is homeomorphic with A''' , A''' is homeomorphic with A'' , and A'' is homeomorphic with $V \subset A' = A$, Φ is contracting on A .

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